DOI: 10.2478/auom-2018-0031
An. Şt. Univ. Ovidius Constanţa
DE GRUYTER
Vol. 26(3),2018, 29–40

Order divisor graphs of finite groups

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Abstract

The interplay between groups and graphs have been the most famous and productive area of algebraic graph theory. In this paper, we introduce and study the graphs whose vertex set is group G such that two distinct vertices a and b having different orders are adjacent provided that $o(a)$ divides $o(b)$ or $o(b)$ divides $o(a)$.

1 Introduction

All finite groups can be represented as the automorphism group of a connected graph [4]. A graphical representation of group can be given by a set of generators and relations. Given any group, symmetrical graphs known as Cayley graphs can be generated, and these have properties related to the structure of the group [5]. Relating a graph to a group provides a method of visualizing a group and connects two important branches of mathematics. It gives a review of cyclic groups, dihedral groups, direct products, generators and relations.

Groups are the main mathematical tools for studying symmetries of an object and symmetries are usually related to graph automorphisms. Many structures in abstract algebra are special cases of groups. Graph theory is one of the leading research field in mathematics mainly because of its applications in diverse fields which include biochemistry, electrical engineering, computer science and operations research. These both branches of mathematics are playing a vital role in modern mathematics. In group theory we study and analyze different groups and their structures while in graph theory we focus

Key Words: Finite groups, Cyclic groups, p-groups, Elementary abelian groups, Connected Graphs, Star Graphs, Sequential Joins.
2010 Mathematics Subject Classification: Primary 05c25.
Received: 5.09.2017.
Accepted: 27.11.2017

on the graphs that denotes the structure of materials and objects. The powerful combinatorial methods found in graph theory have also been used to prove significant and well-known results in a variety of areas in mathematics including group theory.

In the last few decades the researchers have focused on the modified form of groups and graphs by inter relating their properties. The study of the algebraic structures using the properties of graphs has become an inspiring research topic in the last twenty years, leading to many fascinating results and questions, see [2], [1], [7], and [8]. In this paper, we study a graph related to finite groups. We call a graph an *order divisor graph*, denoted by $OD(G)$, if its vertex set is a finite group G and two distinct vertices a and b having different orders are adjacent provided that $o(a)$ divides $o(b)$ or $o(b)$ divides $o(a)$.

For the reader's convenience we give a working introduction here for the notions involved. A group G is called a *p-group* if every element of G has its order a power of p, where p is prime. The *exponent of a group* G is the smallest positive integer m such that $g^m = e$ for all $g \in G$. An abelian group G is called an *elementary abelian group* or *elementary abelian p-group* if it is a p-group of exponent p for some prime p, i.e., $x^p = e \ \ \forall \ x \in G$. A finite *elementary abelian group* is a group that is isomorphic to $\mathbb{Z}_p^n$ for some prime p and for some positive integer n. The set of symmetries of a regular n-gon ($n \geq 3$) forms a group under composition. This group is called the *dihedral group* of order $2n$ and is denoted by D_n. Using the generators and relations, the dihedral group is presented by $D_n =< a, b \mid a^n = b^2 = e, \ ab = ba^{-1} >$ or $D_n =< a, b \mid a^n = b^2 = (ab)^2 = e >$. We denote by $U(\mathbb{Z}_n)$, the group of units of $\mathbb{Z}_n$, i.e., $U(\mathbb{Z}_n) = \{\bar{x} \in \mathbb{Z}_n \mid (x, n) = 1\}$. For any two elements a, b of a group G, $[a, b]$ denotes the *commutator* $a^{-1}b^{-1}ab$. If A, B are subsets of a group G then $[A, B] =< [a, b] \mid a \in A, b \in B >$. Particularly, $[G, G]$ is called *commutator subgroup* denoted by G'. A group G is called *nilpotent group* if and only if it is the direct product of its Sylow subgroups, cf. [6, Chapter 2, Section 3]. The unique maximal nilpotent normal subgroup of a group G is called *Fitting subgroup* of G and is denoted by $F(G)$, cf. [6, Chapter 6, Section 1]. A simple connected graph is an undirected graph without any loops and multiple edges. A complete bipartite graph is a bipartite graph (i.e., a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent) such that every pair of graph vertices in the two sets are adjacent. The star graph S_n of order n is a tree with one vertex of degree $n-1$ and all other vertices have degree 1, i.e., $S_n \cong K_{1,n}$. We denote by $G_1 \diamond G_2 \diamond \cdots \diamond G_k$ the sequential join of graphs $G_1, G_2, ..., G_k$, where $G_i \diamond G_{i+1} = G_i \vee G_{i+1}$ for all $1 \leq i \leq k-1$, i.e., by adding an edge from each vertex of G_i to each vertex of G_{i+1}, $1 \leq i \leq k-1$ The chromatic number

$\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of G, i.e., $\chi(G) = min\{k : G\ is\ k - colorbale\}$.

In this paper we obtain the following results. The order divisor graph $OD(G)$ of a finite group G is a star graph if and only if every non-identity element of G has prime order (Theorem 3). For an abelian group G, $OD(G)$ is a star graph if and only if G is elementary abelian (Corollary 4). The order divisor graph $OD\big(U(\mathbb{Z}_n)\big)$ is a star graph $S_{\phi(n)}$ if and only if $n \mid 24$ (Corollary 6). The order divisor graph $OD(\mathbb{Z}_n)$ is a star graph if and only if n is prime (Corollary 7). The order divisor graph $OD(G)$ of a (finite) group G is a star graph if and only if G is a p-group of exponent p, or a non-nilpotent group of order $p^a q$, or it is isomorphic to the simple group $\mathcal{A}_5$ (Corollary 8). The order divisor graph of the dihedral group D_n $(n \geq 3)$ is a star graph S_{2n} if and only if n is prime (Theorem 9). If G is a finite p-group of order p^n then $OD(G)$ is a complete multi-partite graph (Theorem 10). If G is a finite cyclic group of order p^n then $OD(G)$ is complete $(n+1)$-partite graph (Theorem 12). If G is a finite cyclic group of order p^n, then $\chi(OD(G)) = n + 1$ (Corollary 13). If G is a cyclic group of order p_1p_2, where p_1, p_2 are distinct primes, then $OD(G)$ is a sequential join of graphs (Theorem 15). Similarly, if G is a cyclic group of order $p_1p_2p_3$, where p_1, p_2, p_3 are distinct primes, then $OD(G)$ is obtained by certain type of sequential and cyclic joins. (Theorem 17). Let $n \in \mathbb{N}$ and let D be the set of all (*positive*) divisors of n. Define a partial order $\preceq$ on D by $a \preceq b$ if and only if $a \mid b$. Then $(D, \preceq)$ is a bounded lattice. We denote by G_n the comparability graph on $(D, \preceq)$. In other words, G_n is a simple undirected graph with vertex set D and two vertices a and b are adjacent if and only if $a \neq b$ and either $a \preceq b$ or $b \preceq a$. The new extended graph is denoted by $\mathcal{E}(G_n)$. Given a graph $G = (V, E)$, the reduced graph of G, denoted by $\mathcal{R}(G)$, is obtained from G by merging those vertices which has same set of closed neighbors. Note that a closed neighbor of $v \in V$ is the set $\{u \in V | uv \in E\} \cup \{v\}$. We obtain that if G is finite group of order n, then G is cyclic if and only if $\mathcal{E}(G_n) \cong OD(G)$ if and only if $G_n \cong \mathcal{R}(OD(G))$ (Theorem 21).

In this paper all the groups and graphs discussed are finite. We follow the terminologies and notations of [5] for groups and [9] for graphs.

2 Main results

Definition 1. Let G be a finite group. Then $OD(G)$ denotes the order divisor graph whose vertex set is G such that two distinct vertices a and b having different orders are adjacent provided that $o(a) \mid o(b)$ or $o(b) \mid o(a)$.

Remark 2. Some easy consequences of the definition are:

(i) $OD(G)$ is a simple graphs, so there are no loops and multiple edges.

(ii) Since the identity element is the only element of a group having order one, so the vertex associated to the identity element is adjacent to each vertex and hence $OD(G)$ is always a connected graph. Due to similar reason, if $|G| > 2$, then $OD(G)$ has diameter 2.

(iii) Since the vertex associated to the identity is adjacent to each vertex, therefore, if $|G| > 3$ then $OD(G)$ is not a cycle. If $|G| = 3$ then G has two elements of order 3 and the vertices associated to these two elements are not adjacent. Hence $OD(G)$ cannot be a cycle.

(iv) If G is finite, then for every divisor d of $|G|$, the number of elements of order d is a multiple of $\phi(d)$ (ϕ is Euler's phi function). Hence, if $|G| > 2$, then $OD(G)$ cannot be a complete graph.

Theorem 3. *The order divisor graph $OD(G)$ is a star graph if and only if every non-identity element of the group G has prime order.*

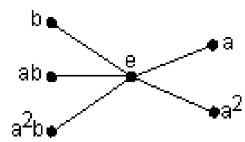
Proof. Clearly, if every non-identity element of G has prime order, then $OD(G)$ is a star graph. Conversely, suppose that $OD(G)$ is a star graph and $x_1, x_2, ..., x_n$ are distinct non-identity elements of G with $o(x_i) = d_i$ for $1 \leq i \leq n$. As $OD(G)$ is a star graph, so $d_i \nmid d_j$ for all $i \neq j$. If d_i is not prime for some i, then d_i has a prime divisor, say p. But then by Cauchy's theorem, G must have an element of order p. Thus $p = d_j$ for some $i \neq j$, which is a contradiction. Hence each d_i is prime. □

Recall [6] that an *elementary abelian group* or *elementary abelian p-group* is an abelian p-group of exponent p for some fixed prime p. A finite *elementary abelian group* is a group that is isomorphic to $\mathbb{Z}_p^n$ for some prime p and for some positive integer n.

Corollary 4. *Let G be an abelian group. Then $OD(G)$ is a star graph if and only if G is elementary abelian.*

Proof. Suppose $OD(G)$ is a star graph. Then by Theorem 3, every non-identity element of G has prime order. Let $a, b \in G$ such that $o(a)$ and $o(b)$ are distinct primes. Since G is abelian, so $o(ab) = o(a)o(b)$, which is a contradiction. Hence G is an elementary abelian group (abelian p-group of exponent p for some prime number p). □

Remark 5. If the group G is not abelian, then above Corollary 4 fails. For example, the order divisor graph $OD(D_3)$ of the dihedral group $D_3 =< a, b \mid a^3 = b^2 = (ab)^2 = e >$ is a star graph but D_3 is not elementary abelian.



$Fig.1 : OD(D_3) \cong S_6$

Corollary 6. *$OD\big(U(\mathbb{Z}_n)\big)$ is a star graph $S_{\phi(n)}$ if and only if $n \mid 24$.*

Proof. Suppose $OD(U(\mathbb{Z}_n)) \cong S_{\phi(n)}$, where ϕ is the Euler's phi function. Then by Corollary 4, $U(\mathbb{Z}_n)$ is an elementary abelian group (abelian p-group of exponent p for some prime number p). Let $\phi(n) > 1, i.e., n > 2$ (If $\phi(n) = 1$, then $U(\mathbb{Z}_n)$ is trivial). Then $U(\mathbb{Z}_n)$ has at least two elements that satisfy $x^2 = e$. Hence every non-identity element of $U(\mathbb{Z}_n)$ has order 2. Case 1: If n is odd. Then $\bar{2} \in U(\mathbb{Z}_n)$ and so $n \mid 2^2 - 1 = 3$. Hence $n = 1$ or 3. Case 2: If $n = 2^t$ for some $t \geq 1$. Then $\bar{3} \in U(\mathbb{Z}_n)$ and so $n \mid 3^2 - 1 = 8$. This implies $n = 1, 2, 4$ or 8. Case 3: If n is any arbitrary positive integer. Then $n = 2^t k$, where k is odd and t is non-negative integer. Since $U(\mathbb{Z}_n) \cong U(\mathbb{Z}_{2^t}) \times U(\mathbb{Z}_k)$, so every non-identity element of both $U(\mathbb{Z}_{2^t})$ and $U(\mathbb{Z}_k)$ has order 2. By above cases 1 and 2, we get that $2^t \in \{1, 2, 4, 8\}$ and $k \in \{1, 3\}$. Hence $n = 2^t k \in \{1, 2, 4, 6, 8, 12, 24\}$.

Conversely, it is easy to check that if $n \mid 24$ then every non-identity element of $U(\mathbb{Z}_n)$ has order 2, i.e., $OD(U(\mathbb{Z}_n)) \cong S_{\phi(n)}$. □

Corollary 7. *$OD(\mathbb{Z}_n)$ is a star graph if and only if n is prime.*

Recall [6] that for any two elements a, b of a group G, $[a, b]$ denotes the *commutator* $a^{-1}b^{-1}ab$. If A, B are subsets of a group G then $[A, B] =< [a, b] \mid a \in A, b \in B >$. Particularly, $[G, G]$ is called *commutator subgroup* denoted by G'. A group G is called *nilpotent group* if and only if it is the direct product of its Sylow subgroups. The unique maximal nilpotent normal subgroup of a group G is called *Fitting subgroup* of G and is denoted by $F(G)$.

Corollary 8. *The order divisor graph $OD(G)$ is a star graph if and only if one of the following cases occur:*

1. *G is p-group of exponent p.*
2. (a) $|G| = p^a q,\ 3 \leq p < q,\ a \geq 3,\ |F(G)| = p^{a-1},\ |G : G'| = p.$

(b) $|G| = p^a q,\ 3 \leq p < q,\ a \geq 1,\ |F(G)| = |G'| = p^a$.

(c) $|G| = 2^a p,\ p \geq 3,\ a \geq 2,\ |F(G)| = |G'| = 2^a$.

(d) $|G| = 2p^a,\ p \geq 3,\ a \geq 1,\ |F(G)| = |G'| = p^a$ *and* $F(G)$ *is elementary abelian.*

3. $G \cong \mathcal{A}_5$.

Proof. Apply [3, Main Theorem]. □

Theorem 9. *The order divisor graph of the dihedral group* D_n $(n \geq 3)$ *is a star graph* S_{2n} *if and only if* n *is prime.*

Proof. $D_n =< a, b \mid a^n = b^2 = (ab)^2 = e >= \{e, a, a^2, ..a^{n-1}, b, ab, a^2b, ..., a^{n-1}b\}$.

Suppose $OD(D_n)$ is a star graph. So, every pair of vertices corresponding to non-identity elements is non-adjacent. Therefore, if $o(a^i) \neq o(a^j)$ for some $i, j \in \{1, 2, 3...n-1\}$, then $o(a^i) \nmid o(a^j)$ and $o(a^j) \nmid o(a^i)$. Note that $o(a^k) = \frac{n}{gcd(k,n)}$ for all $k \in \{1, 2, 3...n-1\}$. Therefore, if $o(a) \neq o(a^k)$ for some $k \in \{1, 2, 3...n-1\}$, then $o(a^k)$ divides $o(a)$ and so a and a^k will become adjacent, which is not possible. Hence $o(a) = o(a^k)$ for all $k \in \{1, 2, 3...n-1\}$. This implies that $gcd(k, n) = 1$ for all $k \in \{1, 2, 3...n-1\}$. Hence n is prime.

Conversely, Suppose that n is prime. Then $o(a^i) = o(a) = n$ for all $i \in \{1, 2, 3...n-1\}$. Moreover, $o(a^i b) = 2$ for all $i \in \{1, 2, 3...n-1\}$. Hence $OD(D_n)$ is a star graph. □

Theorem 10. *Let* G *be a (finite) p-group of order* p^n. *Then order divisor graph* $OD(G)$ *is a complete multi-partite graph.*

Proof. Let p be a prime and $A_i = \{x \in G \mid o(x) = p^i\}$. We can write $G = A_0 \cup A_1 \cup \cdots \cup A_n$. It is not necessary that G has elements of order p^i for $i > 1$ and by Cauchy's theorem, G must have element of order p. Therefore, for some $i > 1$, A_i may be empty but A_0 and A_1 must be nonempty in the union $A_0 \cup A_1 \cup \cdots \cup A_n$. Note that $ab \notin E(OD(G))$ if $a, b \in A_i$ for some $0 \leq i \leq n$. Suppose $a \in A_i$ and $b \in A_j$, where $i \neq j$. Then $o(a) = p^i$ and $o(b) = p^j$. If $i < j$ then p^i strictly divides p^j and if $i > j$ then p^j strictly divides p^i. Therefore $ab \in E(OD(G))$ and hence $OD(G)$ is complete multipartite graph. □

Example 11.

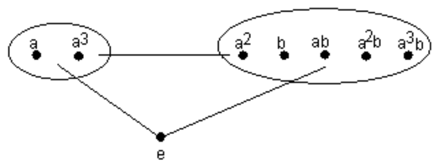


Fig.2: $OD(D_4)$

The line joining two independent sets of vertices means that each vertex in one independent set is adjacent to each vertex of other independent set.

Corollary 12. *Let G be a finite cyclic group of order p^n. Then $OD(G)$ is isomorphic to the complete $(n+1)$-partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{n-1}(p-1)}$.*

Proof. Let $A_i = \{x \in G : o(x) = p^i\}$. We can write $G = A_0 \cup A_1 \cup \cdots \cup A_n$. Since G is cyclic, so by [5, Theorem 4.4], $|A_i| = \phi(p^i) = p^{i-1}(p-1)$, where ϕ is *Euler's phi function.* Note that $ab \notin E(OD(G))$ if $a, b \in A_i$ for any $0 \leq i \leq n$. Suppose $a \in A_i$ and $b \in A_j$, where $i \neq j$. Then $o(a) = p^i$ and $o(b) = p^j$. If $i < j$ then p^i strictly divides p^j and if $i > j$ then p^j strictly divides p^i. Therefore $ab \in E(OD(G))$ and hence $OD(G)$ is isomorphic to the complete $(n+1)$-partite graph $K_{1,p-1,p(p-1),p^2(p-1),\ldots,p^{n-1}(p-1)}$. □

Corollary 13. *If G is a finite cyclic group of order p^n, then $\chi(OD(G)) = n+1$.*

Example 14. The order divisor graph $OD(\mathbb{Z}_8)$ is shown below.

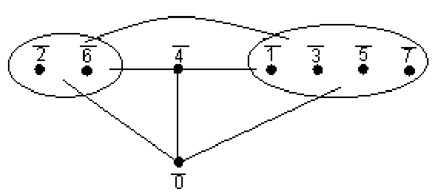


Fig.3: $OD(\mathbb{Z}_8)$

The line joining two independent sets of vertices means that each vertex in one independent set is adjacent to each vertex of other independent set.

We denote by $G_1 \diamond G_2 \diamond \cdots \diamond G_k$ the sequential join of graphs $G_1, G_2, ..., G_k$, where $G_i \diamond G_{i+1} = G_i \vee G_{i+1}$ for all $1 \leq i \leq k-1$, i.e., by adding an edge from each vertex of G_i to each vertex of G_{i+1}, $1 \leq i \leq k-1$.

Theorem 15. *Let G be a cyclic group of order p_1p_2, where p_1, p_2 are distinct primes. Then $OD(G)$ is a sequential join $(G_1 \diamond G_2 \diamond G_3) \diamond K_1$, where $G_1 \cong (p_1-1)K_1$, $G_2 \cong (p_1-1)(p_2-1)K_1$ and $G_3 \cong (p_2-1)K_1$.*

Proof. The divisors of p_1p_2 are $1, p_1, p_2, p_1p_2$. We make a partition of the vertex set G as $G = A_{p_1} \cup A_{p_1p_2} \cup A_{p_2} \cup A_1$, where $A_{p_1} = \{x \in G : o(x) = p_1\}$, $A_{p_1p_2} = \{x \in G : o(x) = p_1p_2\}$, $A_{p_2} = \{x \in G : o(x) = p_2\}$ and $A_1 = \{x \in G : o(x) = 1\}$. By [5, Theorem 4.4], $|A_{p_1}| = \phi(p_1) = p_1 - 1$, $|A_{p_1p_2}| = \phi(p_1p_2) = (p_1-1)(p_2-1)$, $|A_{p_2}| = \phi(p_2) = (p_2-1)$ and $|A_1| = 1$, where ϕ is the Euler's phi function. Since p_1 strictly divides p_1p_2, so each vertex in A_{p_1} is adjacent to each vertex in $A_{p_1p_2}$. Similarly, as p_2 strictly divides p_1p_2, so each vertex in A_{p_2} is adjacent to each vertex in $A_{p_1p_2}$. Note that $p_1 \nmid p_2$ and $p_2 \nmid p_1$, so no vertex of A_{p_1} is adjacent to any vertex of A_{p_2} and no vertex of A_{p_2} is adjacent to any vertex of A_{p_1}. Clearly, the single vertex in A_1, i.e corresponding to the identity element of G, is adjacent to every vertex in A_{p_1}, every element in $A_{p_1p_2}$ and every element in A_{p_2}. Also note that the vertices in A_{p_1}, $A_{p_1p_2}$ and A_{p_2} are independent. Hence $OD(G) = (G_1 + G_2 + G_3) + K_1$, where $G_1 \cong (p_1-1)K_1$, $G_2 \cong (p_1-1)(p_2-1)K_1$ and $G_3 \cong (p_2-1)K_1$. $\square$

Example 16.

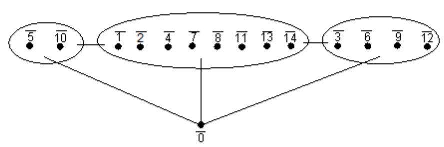


Fig.4: $OD(\mathbb{Z}_{15})$

The line joining two independent sets of vertices means that each vertex in one independent set is adjacent to each vertex of other independent set.

We denote by $G_1 \diamond G_2 \diamond \cdots \diamond G_k$ the sequential join of graphs $G_1, G_2, ..., G_k$, where $G_i \diamond G_{i+1} = G_i \vee G_{i+1}$ for all $1 \le i \le k-1$, i.e., by adding an edge from each vertex of G_i to each vertex of G_{i+1}, $1 \le i \le k-1$.

Theorem 17. *Let H be a cyclic group of order $p_1p_2p_3$, where p_1, p_2 and p_3 are distinct primes. Then $OD(H)$ is defined as:*

$$\Big(\Big((G_1 \diamond G_{12}) \cup (G_{12} \diamond G_2) \cup (G_2 \diamond G_{23}) \cup (G_{23} \diamond G_3) \cup (G_3 \diamond G_{13}) \cup (G_{13} \diamond G_1)\Big) \diamond G_{123}\Big) \diamond K_1$$

where $G_i \cong (p_i - 1)K_1$, $1 \le i \le 3$, $G_{12} \cong (p_1-1)(p_2-1)K_1$, $G_{23} \cong (p_2-1)(p_3-1)K_1$, $G_{13} \cong (p_1-1)(p_3-1)K_1$, and $G_{123} \cong (p_1-1)(p_2-1)(p_3-1)K_1$.

Proof. Let H be a cyclic group such that $|H| = p_1p_2p_3$, where p_1, p_2 and p_3 are distinct primes. The divisors of $p_1p_2p_3$ are $1, p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3, p_1p_2p_3$. We make a partition of the vertex set H, based on the divisors of $p_1p_2p_3$, as follows: $H = A_{p_1} \cup A_{p_1p_2} \cup A_{p_1p_3} \cup A_{p_2} \cup A_{p_2p_3} \cup A_{p_3} \cup A_{p_1p_2p_3} \cup A_1$, where $A_i = \{x \in H \mid o(x) = i\}$ for $i \in \{1, p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3, p_1p_2p_3\}$. By [5, Theorem 4.4], $|A_{p_1}| = \phi(p_1) = p_1 - 1$, $|A_{p_1p_2}| = \phi(p_1p_2) = (p_1 - 1)(p_2 - 1)$, $|A_{p_1p_3}| = \phi(p_1p_3) = (p_1 - 1)(p_3 - 1)$,$|A_{p_2}| = \phi(p_2) = (p_2 - 1)$, $|A_{p_2p_3}| = \phi(p_2p_3) = (p_2 - 1)(p_3 - 1)$, $|A_{p_3}| = \phi(p_3) = (p_3 - 1)$, $|A_{p_1p_2p_3}| = \phi(p_1p_2p_3) = (p_1-1)(p_2-1)(p_3-1)$ and $|A_1| = 1$, where ϕ is the Euler's phi function. Since p_1 strictly divides p_1p_2, p_1p_3 and $p_1p_2p_3$, so each vertex in A_{p_1} is adjacent to each vertex in $A_{p_1p_2}$, $A_{p_1p_3}$ and $A_{p_1p_2p_3}$. As p_2 strictly divides p_1p_2, p_2p_3 and $p_1p_2p_3$, so each vertex in A_{p_2} is adjacent to each vertex in $A_{p_1p_2}$, $A_{p_2p_3}$ and $A_{p_1p_2p_3}$. Similarly p_3 strictly divides p_1p_3, p_2p_3 and $p_1p_2p_3$, so each vertex in A_{p_3} is adjacent to each vertex in $A_{p_1p_3}$, $A_{p_2p_3}$ and $A_{p_1p_2p_3}$. Note that $p_1 \nmid p_2, p_3$, $p_2 \nmid p_1, p_3$ and $p_3 \nmid p_1, p_2$, so no vertex of A_{p_1} is adjacent to any vertex of A_{p_2} or A_{p_3}, no vertex of A_{p_2} is adjacent to any vertex of A_{p_1} or A_{p_3} and similarly no vertex of A_{p_3} is adjacent to any vertex of A_{p_1} or A_{p_2} . Clearly, the single vertex in A_1, i.e., corresponding to the identity element of H, is adjacent to every vertex in A_{p_1}, A_{p_2}, A_{p_3}, every vertex in $A_{p_1p_2}$, $A_{p_1p_3}$, $A_{p_2p_3}$ and every vertex in $A_{p_1p_2p_3}$. Also note that the sets of vertices A_{p_1}, A_{p_2}, A_{p_3}, $A_{p_1p_2}$, $A_{p_1p_3}$, $A_{p_2p_3}$ and $A_{p_1p_2p_3}$ are independent sets. Hence $OD(H)$ is defined as $\Bigg(\Big((G_1 \diamond G_{12}) \cup (G_{12} \diamond G_2) \cup (G_2 \diamond G_{23}) \cup (G_{23} \diamond G_3) \cup (G_3 \diamond G_{13}) \cup (G_{13} \diamond G_1)\Big) \diamond G_{123}\Bigg) \diamond K_1$. □

Example 18.

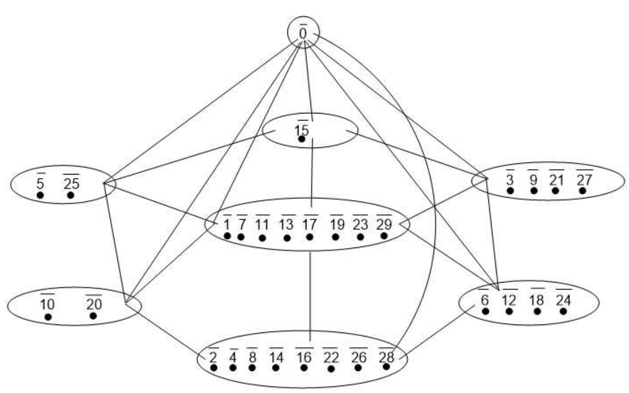


Fig.5: $OD(\mathbb{Z}_{30})$

The line joining two independent sets of vertices means that each vertex in one independent set is adjacent to each vertex of other independent set.

Definition 19. Let $n \in \mathbb{N}$ and let D be the set of all (*positive*) divisors of n. Define a partial order $\preceq$ on D by $a \preceq b$ if and only if $a \mid b$. Then $(D, \preceq)$ is a bounded lattice. We denote by G_n the comparability graph on $(D, \preceq)$. In other words, G_n is a simple undirected graph with vertex set D and two vertices a and b are adjacent if and only if $a \neq b$ and either $a \preceq b$ or $b \preceq a$. We denote by $\mathcal{E}(G_n)$ the extended graph of G_n which is obtained by replacing each vertex d in G_n by $\phi(d)$ copies of d which form an independent set.

Definition 20. Given a graph $G = (V, E)$, the reduced graph of G, denoted by $\mathcal{R}(G)$, is obtained from G by merging those vertices which has same set of closed neighbors, where a closed neighbor of $v \in V$ is the set $\{u \in V | uv \in E\} \cup \{v\}$.

Theorem 21. *Let G be a finite group of order n. The following are equivalent:*

(a) *G is cyclic;*

(b) *$\mathcal{E}(G_n) \cong OD(G)$;*

(c) *$G_n \cong \mathcal{R}(OD(G))$.*

Proof. $(a) \Leftrightarrow (b)$: Let D be the set of all (*positive*) divisors of n. Suppose G is cyclic. Then for each $d \in D$, G has exactly $\phi(d)$ elements of order d, cf. [5, Theorem 4.4]. Hence by definitions of $OD(G)$ and G_n, we get that $E(G_n) \cong OD(G)$. Conversely, suppose $\mathcal{E}(G_n) \cong OD(G)$. Since $n \in D$, so G_n has a vertex associated to n and hence $\mathcal{E}(G_n)$ has $\phi(n)$ vertices associated to the group elements of order n. Hence G is cyclic.

$(a) \Leftrightarrow (c)$: Let D be the set of all distinct (*positive*) divisors of n. Suppose G is cyclic. Then for each $d \in D$, G has exactly $\phi(d)$ elements of order d, cf. [5, Theorem 4.4]. Therefore, by definitions, G_n is isomorphic to the reduced graph of $OD(G)$. Conversely, let $G_n \cong \mathcal{R}(OD(G))$. Since, $OD(G)$ consists of independent sets associated to elements of same order in G and by reducing $OD(G)$ we obtain the comparability graph G_n (in G_n we have a vertex associated to each divisor of n), therefore in $OD(G)$ we have independent set of vertices associated to each divisor of $|G| = n$. In particular, G must have elements or order n. Hence G is cyclic.

□

Remark 22. We saw that for finite cyclic groups G, the comparability graph G_n can be studied by passing to the order divisor graph $OD(G)$. Also the order divisor graph $OD(G)$ can be studied by passing to the comparability graph G_n. That is, we have maps

$$\{Comparability\ graphs\} \underset{\mathcal{R}}{\overset{\mathcal{E}}{\rightleftharpoons}} \{Order\ divisor\ graphs\}$$

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